Name: ______________________________

Teacher: ____________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2011

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 2

# Mathematics Extension 1

TIME ALLOWED: 1½ HOURS

| Outcomes Assessed | Questions | Marks |
|---|---|---|
| Manipulates algebraic expressions to solve problems involving inverse functions | 1, 2, 4 | |
| Applies appropriate techniques to solve problems involving parametric representations | 3, 5 | |

| Question | 1 | 2 | 3 | 4 | 5 | Total | % |
|---|---|---|---|---|---|---|---|
| Marks | /10 | /11 | /9 | /12 | /10 | /52 | |

## Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- **Each new question is to be started in a new booklet**

# STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

**Question 1 (10 marks)**

a) Differentiate with respect to $x$ : $y = cos^{-1}(3x-1)$. **1**

b) Find the primitive function of

i) $\dfrac{1}{1+3x^2}$ **1**

ii) $\dfrac{x+2}{x^2+4}$ **2**

c) Evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx$ **2**

d) Without evaluating the integral explain why $\displaystyle\int_{\frac{-1}{2}}^{\frac{1}{2}} sin^{-1} x \quad dx$ is equal to zero. **1**

e) Show that $tan^{-1}x = cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$ . **3**

**Question 2 (11 marks)**

a) Consider the function $y = 2cos^{-1}\dfrac{x}{3}$ .

i) Sketch the graph of this function clearly showing the domain and range. **2**

ii) Find the angle $\theta$, that the tangent to the curve $y = 2\cos^{-1}\dfrac{x}{3}$ at $x = 0$ makes with the positive direction of the $x$ axis. Give your answer in degrees and minutes. **3**

iii) Find the area between this curve and the axes in the first quadrant. **3**

b) Sketch $y = \tan^{-1}(\sin 3x)$ for $\dfrac{-\pi}{3} \le x \le \dfrac{\pi}{3}$, including any stationary points. **3**

**Question 3 (9 marks)**

a) The tangent at $P\left(2ap, ap^2\right)$ on the parabola $x^2 = 4ay$ meets the $x$ axis at $T$. The normal at $P$ meets the $y$ axis at $N$.

i) Show that the equation of the normal at $P$ is $x + py = 2ap + ap^3$. **1**

ii) Find the co-ordinates of $M$, the midpoint of $TN$. **2**

iii) Show that the locus of $M$ is the parabola $x^2 = \frac{a}{2}(y-2)$. **2**

b) The point $A(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$.

i) Given that the normal at A passes through the point $R(-6a, 9a)$ show that $p^3 - 7p + 6 = 0$. **1**

ii) Hence, find the values of $p$ on this parabola at which the normals intersect at $R$. **3**

**Question 4 (12 marks)**

a) Find the exact equation (in general form) of the tangent to the curve $y = \sin^{-1}\sqrt{x}$ at the point where $x = \frac{1}{2}$. **3**

b)

i) Differentiate $x\, cos^{-1}x - \sqrt{1-x^2}$ with respect to $x$. **2**

ii) Hence evaluate $\int_0^1 \cos^{-1} x\, dx$. **2**

c) The area bounded by the curve $y = \frac{1}{\sqrt[4]{4-x^2}}$, the $x$ and $y$ axes and the lines $x = 0$ and $x = \sqrt{3}$ is rotated about the $x$ axis. Find the volume generated in exact form. **2**

d) Find showing all necessary working, the exact value of $\tan\left(2\sin^{-1}\left(\frac{-1}{4}\right)\right)$. **3**

**Question 5 (10 marks)**

a) What is the Cartesian equation for the parametric equations $x = t - 2, y = t^2 - 4$? **1**

b) Tangents from the point $P(x_{0,}y_0)$ touch the parabola $x^2 = 4y$ at $Q$ and $R$.

i) Prove that the midpoint $T$ of $QR$ is $\left(\mathrm{x}_0, \frac{1}{2}(x_0)^2 - \mathrm{y}_0\right)$. **5**

ii) If P moves on the line $x - y = 1$, find the equation of the locus of $T$. **3**

iii) Describe this locus geometrically. **1**

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EXT. I HSC ASSESSMENT (2) | Comments

Question 1

a) $y = \cos^{-1}(3x-1)$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(3x-1)^2}} \times 3$

$= \frac{-3}{\sqrt{1-(9x^2-6x+1)}}$

$\frac{dy}{dx} = \frac{-3}{\sqrt{6x-9x^2}}$ ✓

many forgot to multiply by 3. ie not using function of a function rule correctly. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

b) i) $\int \frac{1}{1+3x^2}\,dx$

$= \frac{1}{\sqrt{3}} \int \frac{1 \times \sqrt{3}}{1+(\sqrt{3}x)^2}\,dx$

$= \frac{1}{\sqrt{3}} \tan^{-1}\sqrt{3}x + c$ ✓

mostly well done.

ii) $\int \frac{x+2}{x^2+4}\,dx$

$= \int \frac{x}{x^2+4}\,dx + \int \frac{2}{x^2+4}\,dx$ ✓

✓

$= \frac{1}{2}\ln(x^2+4) + \tan^{-1}\frac{x}{2} + c$

c) $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{2^2-x^2}}\,dx$

mostly well done.

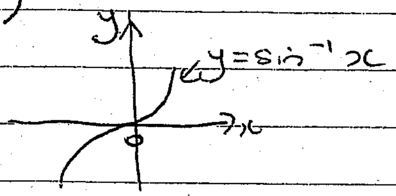
$= \left[\sin^{-1}\frac{x}{2}\right]_1^{\sqrt{3}}$ ✓

$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{3} - \frac{\pi}{6}$

$= \frac{\pi}{6}.$ ✓

Q1 d)




As $y=\sin^{-1}x$ is an odd fn it has pt. symmetry ie. the area above the $x$ equals the area below the axis & $\therefore$ cancels each other out.

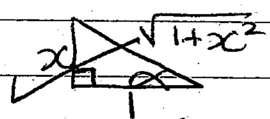
many did not explain clearly how the area below and above the $x$-axis cancels each other out.

e) Show that $\tan^{-1}x = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$

Mostly well done

Let $\tan^{-1}x = \alpha$

$\therefore \quad \dfrac{x}{1} = \tan\alpha$




$\therefore \cos\alpha = \dfrac{1}{\sqrt{1+x^2}}$ (using rt. Δ)

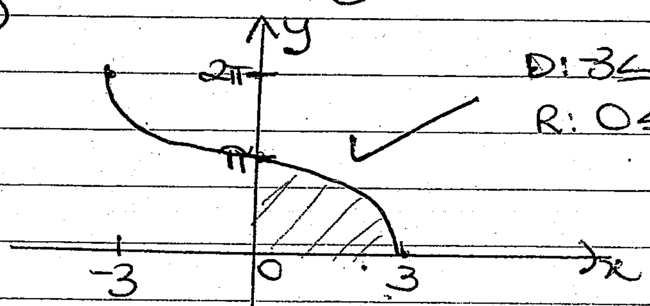
$\therefore \alpha = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$

but $\alpha = \tan^{-1}x$

$\therefore \tan^{-1}x = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$

**QUESTION 2**

a) i) $y = 2\cos^{-1}\dfrac{x}{3}$




D: $-3 \le x \le 3$
R: $0 \le y \le 2\pi$

mostly well done

ii) $f(x) = 2\cos^{-1}\dfrac{x}{3}$

$f'(x) = 2 \cdot \dfrac{-1}{\sqrt{9-x^2}}$

$= \dfrac{-2}{\sqrt{9-x^2}}$

$f'(0) = -\dfrac{2}{3}$

$\therefore \tan\theta = -\dfrac{2}{3}$

$\theta = 146°19'$ (nearest min)

poorly attempted
many did not make the connections between $f'(x) = m = \tan\theta$

Q2 (cont'd)

2a) iii) $A = \int 2\cos^{-1}\frac{x}{3}\,dx.$

$$y = 2\cos^{-1}\frac{x}{3}$$

$$\frac{y}{2} = \cos^{-1}\frac{x}{3}$$

$$3\cos\frac{y}{2} = x$$

$\therefore A = 3\int_0^{\pi} \cos\frac{y}{2}\,dy$ ✓

$= 3\times 2\left[\sin\frac{y}{2}\right]_0^{\pi}$ ✓

$= 6\left[\sin\frac{\pi}{2} - \sin 0\right]$

$= 6(1-0)$ ✓

$A = 6\ u^2.$

some tried to integrate this directly rather than finding the area bounded by the curve and y-axis

b) $y = \tan^{-1}(\sin 3x)$

$\frac{dy}{dx} = \frac{1}{1+\sin^2 3x}\cdot 3\cos 3x$

for stat$^{ny}$ pts $\frac{dy}{dx} = 0.$

$\therefore 3\cos 3x = 0$

$\cos 3x = 0$

$3x = -\frac{\pi}{2}, \frac{\pi}{2}$

$x = -\frac{\pi}{6}, \frac{\pi}{6}$ ✓

when $x = \frac{\pi}{6}$:

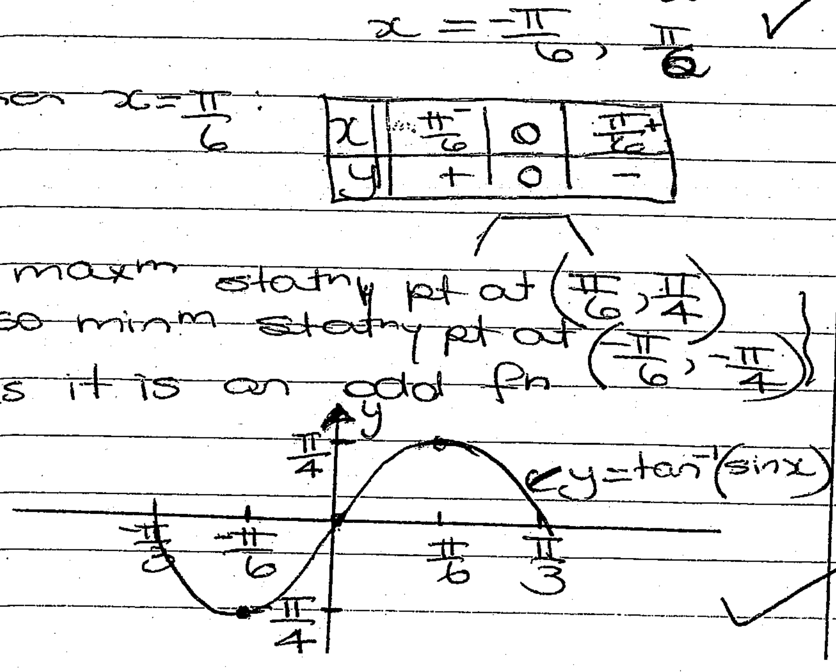
| $x$ | $\frac{\pi}{6}^-$ | 0 | $\frac{\pi}{6}^+$ |
|---|---|---|---|
| $y'$ | + | 0 | − |

$\therefore$ max$^m$ stat$^{ny}$ pt at $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

also min$^m$ stat$^{ny}$ pt at $\left(-\frac{\pi}{6}, -\frac{\pi}{4}\right)$

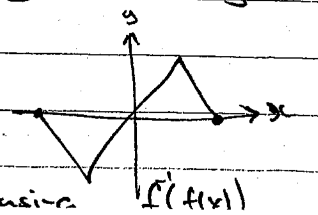
as it is an odd fn

many did not confirm/test the nature of the stationary points




✓

many incorrectly had




confusing $f'(f(x))$

QUESTION 3.

a) $P(2ap, ap^2)$ $x^2 = 4ay$

i) $y = \frac{x^2}{4a}$

$y' = \frac{x}{2a}$

at $x = 2ap$ $y' = p$

$\therefore$ grad. normal $= -\frac{1}{p}$

eqn of normal: $y - ap^2 = -\frac{1}{p}(x - 2ap)$ ✓

$py - ap^3 = -x + 2ap$

$\therefore x + py = 2ap + ap^3$ (as reqd)

well done.

ii) Eqn of tangent is:

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

at T $y = 0 \Rightarrow px = ap^2$

$x = ap$

$\therefore T(ap, 0)$ ✓

$N(0, 2a + ap^2)$

$\therefore M\left(\frac{ap}{2}, \frac{2a + ap^2}{2}\right)$

$\Rightarrow M\left(\frac{ap}{2}, a + \frac{1}{2}ap^2\right)$ ✓

- students need to find T and N to get one mark.
- some mixed up when to let $x = 0$, & $y = 0$, which resulted in error carried for pt M & made it much harder.

iii) $x = \frac{ap}{2}$

$\therefore p = \frac{2x}{a}$

$y = a + \frac{1}{2}ap^2$

$2y = 2a + a\left(\frac{2x}{a}\right)^2$ ✓

$= 2a + \frac{4x^2}{a}$

$2ay = 2a^2 + 4x^2$

$4x^2 = 2ay - 2a^2$ is the locus of M

$x^2 = \frac{\cancel{2}ay}{\cancel{4}\,2} - \frac{\cancel{2}a^2}{\cancel{4}\,2}$ ✓

$x^2 = \frac{a}{2}(y - a)$

many ignored the 'typo' & got correct answer.
Others had correct answer & just changed it to a 2 for convenience.
- Marked very leniently
- Others totally ...

Q3 b) $A(2ap, ap^2)$ $\quad x^2 = 4ay$

i) normal is: $x + py = 2ap + ap^3$

thro' $(-6a, 9a)$: $-6a + 9ap = 2ap + ap^3$

$-6 + 7p = p^3$

$\therefore p^3 - 7p + 6 = 0$ (as req'd)

No need to derive formula again, just state it.

ii) Let $f(p) = p^3 - 7p + 6$

$f(1) = 0$

$\therefore (p-1)$ is a factor

many had trouble finding any factor first & hence made it hard to complete.

$$\begin{array}{r} p^2 + p - 6 \\ (p-1)\overline{)\,p^3 - 7p + 6} \\ \underline{p^3 - p^2} \\ p^2 - 7p + 6 \\ \underline{p^2 - p} \\ -6p + 6 \\ \underline{-6p + 6} \\ 0 \end{array}$$

$\therefore p^3 + p - 6 = (p-1)(p^2 + p - 6)$

$= (p-1)(p-2)(p+3)$

$\therefore p = 1, 2, -3.$

QUESTION 4:

a) $y = \sin^{-1}\sqrt{x}$

$y' = \dfrac{1}{\sqrt{1 - \sqrt{x}^2}} \cdot \dfrac{1}{2}x^{-\frac{1}{2}}$

• many did not use the chain rule correctly here (yet did so in pt(b)).

$= \dfrac{1}{2\sqrt{x(1-x)}}$

$= \dfrac{1}{2\sqrt{x - x^2}}$

when $x = \frac{1}{2}$, $y' = \dfrac{1}{2\sqrt{\frac{1}{4}}}$

$\therefore m = 1$

when $x = \frac{1}{2}$ $\quad y = \sin^{-1}\frac{1}{\sqrt{2}}$

$y = \frac{\pi}{4}$ $\quad \Rightarrow \left(\frac{1}{2}, \frac{\pi}{4}\right)$

both

$\therefore y - \frac{\pi}{4} = 1\left(x - \frac{1}{2}\right)$

$4y - \pi = 4x - 2$

$4x - 4y - 2 + \pi = 0$ is the tangent.

• general form has no fractions in it!

4(b) i) let $y = \underset{u}{x}\,\underset{v}{\cos^{-1}x} - \sqrt{1-x^2}$

$y' = \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} \; -\frac{1}{\cancel{2}}(1-x^2)^{-\frac{1}{2}}\cdot -\cancel{2}x$

$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$

$y' = \cos^{-1}x$

- many had problems with signs, so did not eliminate terms.

ii) $\int_0^1 \cos^{-1}x \, dx$

$= \left[\boxed{x\cos^{-1}x - \sqrt{1-x^2}}\right]_0^1$

$= (\cos^{-1}1 - 0) - (0 - 1)$

$= 0 + 1$

$= 1$

- mostly well done - many got -1 as an answer because they didn't use brackets appropriately!

(c) $V = \pi \int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx$

$= \pi \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}$

$= \pi \left[\sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}0\right)$

$= \pi\left(\frac{\pi}{3} - 0\right)$

$V = \frac{\pi^2}{3}\,u^3$

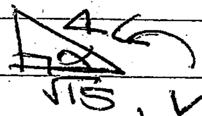
- Some strange variants on this standard integration - use the "Tables of Standard Integral" provided!

(d) $\tan\left(2\sin^{-1}\left(-\frac{1}{4}\right)\right)$

$= -\tan\left(2\sin^{-1}\frac{1}{4}\right)$

let $\sin^{-1}\frac{1}{4} = \alpha$

$\therefore \sin\alpha = \frac{1}{4}$



now $\tan\left(2\sin^{-1}\left(-\frac{1}{4}\right)\right) = -\tan 2\alpha$

$= \frac{-2\tan\alpha}{1-\tan^2\alpha}$

$= \frac{-2\times\frac{1}{\sqrt{15}}}{1-\frac{1}{15}}$

$= \frac{-2}{\sqrt{15}} \times \frac{15}{14}$

$= \frac{-15}{7\sqrt{15}} \times \frac{\sqrt{15}}{\sqrt{15}}$

$\tan\left(2\sin^{-1}\left(-\frac{1}{4}\right)\right) = \frac{-\sqrt{15}}{7}$

- many errors here, usually related to not understanding the restricted domain/range when dealing with inverse trig functions!
- Also simple trig identity errors with
  $\sin(-x) = -\sin x$
  $\tan(-x) = -\tan x$
- many had $1 + \tan^2\alpha$ for their denominator!

## Question 5:

a) $x = t - 2 \qquad y = t^2 - 4$

$t = x + 2 \qquad y = (x+2)^2 - 4$

$= x^2 + 4x + \cancel{4} - \cancel{4}$

$\boxed{y = x^2 + 4x}$ ✓

To obtain mark students had to have in simplified form.

b) $x^2 = 4ay$

$\Rightarrow x^2 = 4y \quad \therefore \quad a = 1$

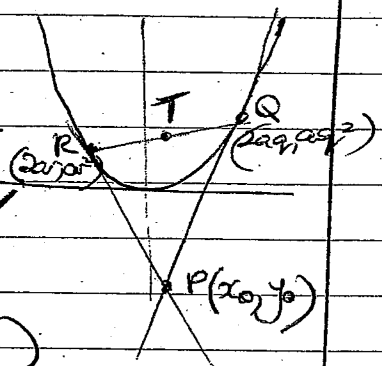
Let R be $(2ar, ar^2) \Rightarrow R(2r, r^2)$

Q be $(2aq, aq^2) \Rightarrow Q(2q, q^2)$

$m_{RQ} = \dfrac{r^2 - q^2}{2r - 2q}$

$= \dfrac{\cancel{(r-q)}(r+q)}{2\cancel{(r-q)}}$

$m_{RQ} = \dfrac{r+q}{2}$ ✓




Students attempted various ways of doing this and often got into a real mess

eqⁿ QR:

$y - q^2 = \dfrac{r+q}{2}(x - 2q)$

$y - q^2 = \dfrac{(r+q)}{2}x - q(r+q)$

$y - \cancel{q^2} = \dfrac{(r+q)}{2}x - \cancel{q^2} - rq$

$2y + 2qr = (r+q)x$

$x(r+q) = 2(y + qr)$ ✓ ... (1)

Now eqⁿ of chord of contact from external pt is $xx_0 = 2a(y + y_0)$ ✓ (2)

ie $xx_0 = 2(y + y_0)$

comparing (1) + (2)

$x_0 = r + q$ and $y_0 = qr$ ✓

Now midpt of QR: $\left(q + r, \dfrac{r^2 + q^2}{2}\right)$

$= \left(q + r, \dfrac{(r+q)^2 - 2rq}{2}\right)$ ✓

$= \left(x_0, \dfrac{1}{2}(x_0)^2 - y_0\right)$ as req'd.

ii) If P moves on $x-y=1$

$y=x-1$

then $y_0=x_0-1$

Locus of T: $x=x_0;\ y=\frac{1}{2}(x_0)^2-y_0$

$=\frac{1}{2}(x_0)^2-(x_0-1)$

$=\frac{1}{2}x^2-(x-1)$

$=\frac{1}{2}x^2-x+1$

$=\frac{1}{2}(x^2-2x+1)+\frac{1}{2}$

$y=\frac{1}{2}(x-1)^2+\frac{1}{2}$

$y-\frac{1}{2}=\frac{1}{2}(x-1)^2$

$2y-1$

$\therefore$ Locus is $\boxed{(x-1)^2=2(y-\frac{1}{2})}$

iii) this is a parabola (concave up)

$V(1,\frac{1}{2})$ focal length $=\frac{1}{2}$ unit.

Need to give more information other than it is a parabola